A DEGREE THEORY FRAMEWORK FOR SEMILINEAR ELLIPTIC SYSTEMS

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Abstract. This paper establishes the existence of positive entire solutions to some systems of semilinear elliptic equations including those related to the Lane-Emden and stationary Schrödinger systems. The primary technique for generating these results employs a degree theoretic approach for the classical shooting method.

1. Introduction and main results

In this article, we establish the existence of positive bound states for a class of semilinear systems of the form

$$\begin{cases} -\Delta u_i = f_i(u_1, u_2, \ldots, u_L) & \text{in } \mathbb{R}^n, \\ u_i > 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $i = 1, 2, \ldots, L$ and $n \geq 3$. Namely, we determine the suitable conditions on system (1.1) that guarantee the existence of bounded, non-trivial classical solutions which decay and vanish at infinity. We do so by using a framework that employs a degree theory approach for the classical shooting method (we refer the reader to [20] and [29] for methods utilizing similar ideas). Consequently, the examination of this general class of systems allows us to obtain existence results for systems of the Lane-Emden and stationary Schrödinger types. In particular, we obtain results for systems of the Schrödinger type

$$\begin{cases} -\Delta u = u^s v^q & \text{in } \mathbb{R}^n, \\ -\Delta v = v^t u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

and to a related system with weakly coupled nonlinearities

$$\begin{cases} -\Delta u = u^s + v^q & \text{in } \mathbb{R}^n, \\ -\Delta v = v^t + u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1.3}$$

(see [10, 25, 30]) where $s, t, p, q \geq 0$. These systems have garnered some recent attention and they constitute the central and motivating examples of this paper. In view of this, let us discuss several known and related results for the Lane-Emden and Schrödinger type systems. If $s = t$ and $p = q$ in (1.2), we arrive at the system

$$\begin{cases} -\Delta u = u^s v^p & \text{in } \mathbb{R}^n, \\ -\Delta v = v^s u^p & \text{in } \mathbb{R}^n, \end{cases} \tag{1.4}$$

which is closely related to the stationary Schrödinger systems for the Bose-Einstein condensate (see [18, 19]). In the special case when $n = 3$, $s = 2$, $p = 3$, and $u \equiv v$,

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system (1.4) reduces to the quintic stationary Schrödinger equation considered by Bourgain in [1]. In fact, when $1 \leq s < p \leq \frac{n+2}{n-2}$ with $s+p = \frac{n+2}{n-2}$, the radial symmetry, monotonicity, and uniqueness (i.e. symmetry of components) of positive bound states to (1.4) was examined by Li and Ma in [14]. These results were later extended, among other things, by Quittner and Souplet in [25]. Particularly, one of the main results of [14] determined the conditions on system (1.4) and its solutions which guaranteed that $u \equiv v$. That is, the system actually reduces to the scalar equation

$$-\Delta u = u^{\frac{n+2}{n-2}} \text{ in } \mathbb{R}^n, \tag{1.5}$$

and therefore the celebrated classification result for equation (1.5) [3, 6, 11] implies that

$$u(x) = v(x) = c_n\Big(\frac{t}{t^2+|x-x_0|^2}\Big)^{\frac{n-2}{2}}, \tag{1.6}$$

for some constants $c_n, t > 0$ and some point $x_0 \in \mathbb{R}^n$.

Another noteworthy case of system (1.2) is the well-known Lane-Emden system

$$\begin{cases} -\Delta u = v^q, & u > 0 \quad \text{in } \mathbb{R}^n, \\ -\Delta v = u^p, & v > 0 \quad \text{in } \mathbb{R}^n, \end{cases} \tag{1.7}$$

which has attracted much attention, especially on questions pertaining to existence and non-existence of classical solutions. In particular, it is conjectured that the Lane-Emden system has no classical solution if and only if the subcritical case holds, $\frac{1}{1+p} + \frac{1}{1+q} > \frac{n-2}{n}$. This is commonly referred to as the Lane-Emden conjecture. Up to now this conjecture has been resolved for radial solutions (see [7, 22]), for dimension $n \leq 4$ (see [24, 27, 28]), and for $n \geq 5$ but only under subregions of the sub-critical Sobolev hyperbola (see [2, 9, 22, 26]). It is well known that such Liouville type theorems are key to establishing singularity analysis and a priori estimates for solutions to the Dirichlet problem for a family of second-order elliptic equations. We should also add that the Lane-Emden system and its scalar counterpart (1.5) are closely related to the Yamabe and prescribing scalar curvature problems (see [3, 4, 5, 12, 13, 17] and the references therein) and they are also connected with finding the best constant in the Hardy-Littlewood-Sobolev inequality (see [16] and references therein).

We emphasize that our results not only apply to the Lane-Emden system, but also for systems without variational structure (e.g. the Schrödinger systems), and we do so by incorporating only basic tools. It is also worth noting that our methods here apply to a more general family of elliptic systems not included in past works. Our main existence results are as follows. Here we call a positive solution (u, v) a **bound state** if it is bounded, radially symmetric and $u, v \longrightarrow 0$ uniformly as $|x| \longrightarrow \infty$.

Theorem 1. *Let $q \geq t > 1$ and $p \geq s > 1$.*

(a) (Existence) System (1.2) admits a positive bound state solution if either

$$(i)\ \frac{1}{1+p} + \frac{1}{1+q} \leq \frac{n-2}{n} \ \text{ or } \ (ii)\ \min\{s+q, t+p\} \geq \frac{n+2}{n-2}.$$

(b) (Symmetry) Additionally, if

$$q + s = p + t, \tag{1.8}$$

then the solution components are symmetric, i.e., $u \equiv v$. *Therefore if*

$$p + t = q + s = (n + 2)/(n - 2),$$

then the solution is the essentially unique solution, i.e., it assumes the form (1.6).

Remark 1. In contrast with the Lane-Emden system, a complete understanding of the optimal conditions for the existence and non-existence of positive solutions for system (1.2) is still missing. However, Theorem 1 does provide two separate conditions where one does not properly contain the other, and vice versa. For instance, if $n = 3$, $s = t = 2$ and $p = q = 3$, then (ii) is satisfied but (i) is not. Conversely, if $n = 3$, $s > 1$ and $p = 3$, then we can choose large $q \geq 11$ and $t \in (1, 2)$ so that (i) is satisfied but not (ii).

Theorem 2. *System* (1.3) *admits a positive bound state solution whenever*

$$\min\{s, t, p, q\} \geq (n + 2)/(n - 2).$$

The rest of this article is structured as follows. Section 2 introduces some notation and preliminary background and states an existence result for the general system (1.1). Section 3 describes our degree theoretic framework then proves our existence result for system (1.1). Section 4 recalls some non-existence results for related boundary value problems required in the proofs of Theorems 1 and 2, which are then given at the conclusion of the section.

2. Preliminaries and a general result

Let $\mathbb{R}^L_+$ be the L-times Cartesian product of the interval $\mathbb{R}_+ = [0, \infty)$. Consider

$$\text{(Global)} \qquad \begin{cases} -\Delta u_i = f_i(u_1, u_2, \ldots, u_L) & \text{in } \mathbb{R}^n, \\ u_i > 0 & \text{in } \mathbb{R}^n, \\ u_i \longrightarrow 0 \text{ uniformly as } |x| \longrightarrow \infty, & \text{for } i = 1, 2, \ldots, L, \end{cases}$$

where $F = (f_1, f_2, \ldots, f_L) : \mathbb{R}^L_+ \longrightarrow \mathbb{R}^L_+$ is a continuous vector-valued map locally Lipschitz continuous in $int(\mathbb{R}^L_+)$, the interior of $\mathbb{R}^L_+$, and $F(0) = 0$.

Further Notation: Throughout, we let $B_R(x) \subset \mathbb{R}^n$ be the open ball of radius R centered at x with boundary $\partial B_R(x)$. At the expense of slightly abusing notation, we denote vector solutions $(u_1, u_2, \ldots, u_L) \in \mathbb{R}^L_+$ simply by u when dealing with system (Global) whereas u and v are understood to be scalar-valued functions when dealing with systems (1.2) and (1.3). We say that $u > 0$ (resp. $u = 0$) in $\mathbb{R}^L_+$ if $u_i > 0$ (resp. $u_i = 0$) for $i = 1, 2, \ldots, L$. For any permutation $\tilde{I} = \{i_1, i_2, \ldots, i_L\}$ of the set $I = \{1, 2, \ldots, L\}$, any positive pair of reals $M > m$ and integer $1 \leq j < L$, we define $\Omega = \Omega(\tilde{I}, m, M, j) \subset \mathbb{R}^L_+$ to be the subset

$$\Big\{ v \in \mathbb{R}^L_+ \,\Big|\, v_{i_k} \leq m,\ k = 1, 2, \ldots, j;\ m < v_{i_k} \leq M,\ k = j + 1, j + 2, \ldots, L \Big\}.$$

In addition, we shall always assume hereafter that system (Global) satisfies some **non-degeneracy** conditions; namely,

(a) $F(v) = 0$ implies that $v \in \partial\mathbb{R}^L_+$.

(b) For each pair $M > m > 0$, there exists a positive constant $C_{m,M}$, depending only on m and M, such that

$$\sum_{k=j+1}^{L} f_{i_k}(v) \leq C_{m,M} \left\{ \frac{1}{L} \sum_{k=1}^{j} f_{i_k}(v) \right\} \text{ in } \Omega \tag{2.1}$$

for all permutations $\tilde{I}$ of I and all integers $1 \leq j < L$.

Remark 2. Our non-degeneracy conditions are quite important and we shall see their precise roles in the next section. Particularly, these assumptions guarantee that the target map we construct shortly below will be continuous–an important property needed in our degree theoretic method. Meanwhile, the assumptions also ensure we avoid trivial solutions or the semi-trivial ones of the form $(u, 0)$ or $(0, v)$.

Remark 3. It is not too difficult to verify that systems (1.2) and (1.3), under the conditions indicated in our main theorems, satisfy the non-degeneracy conditions.

Throughout, the vector-valued solutions $u = (u_1, u_2, \ldots, u_L)$ of system (Global) are to be understood in the classical sense, i.e., $u_i \in C^2(\mathbb{R}^n)$ for $i = 1, 2, \ldots, L$. Likewise, when considering its corresponding boundary value problem,

$$\begin{cases} -\Delta u_i = f_i(u_1, u_2, \ldots, u_L) & \text{in } B_R(0), \\ u_i > 0 & \text{in } B_R(0), \\ u_i = 0 & \text{on } \partial B_R(0), \text{ for } i = 1, 2, \ldots, L, \end{cases} \tag{Local}$$

the solution components are understood to be of the class $C^2(B_R(0)) \cap C^1(\overline{B_R(0)})$.

There holds the following general existence result.

Theorem 3. *System* (Global) *admits a radially symmetric solution provided that the corresponding boundary value problem* (Local) *admits no radially symmetric solution for any* $R > 0$.

Remark 4. In view of Theorem 3 and some known non-existence results for boundary value problems corresponding to our motivating examples, we will establish the existence results for the Schrödinger and Lane-Emden type systems.

3. Proof of Theorem 3

To establish our general result, we introduce several important ideas. The proof relies on constructing a particular map closely related to the shooting method. We begin here by defining this map. For any positive initial value $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_L)$, i.e., $\alpha_i > 0$ for each $i = 1, 2, \ldots, L$, consider the initial value problem,

$$\begin{cases} -\left(u_i''(r) + \dfrac{n-1}{r} u_i'(r) \right) = f_i(u(r)) \\ u_i'(0) = 0, \ u_i(0) = \alpha_i \text{ for } i = 1, 2, \ldots, L. \end{cases} \tag{3.1}$$

Definition 1. Define the target map $\psi : \mathbb{R}^L_+ \longrightarrow \partial \mathbb{R}^L_+$ as follows. For α belonging to $int(\mathbb{R}^L_+)$, we set

(i) $\psi(\alpha) := u(r_0)$ where r_0 is the smallest finite value of r for which $u_{i_0}(r) = 0$ for some $1 \leq i_0 \leq L$. We say that this is the case where the solution touches the wall.

(ii) If no such finite r_0 exists, then we take $r_0 = \infty$ and set $\psi(\alpha) := \lim_{r \to r_0} u(r)$. We say that this is the case where the solution never touches the wall.

Moreover, we define $\psi \equiv Identity$ on the boundary $\partial\mathbb{R}^L_+$.

Remark 5. Note that we are making use of the monotonicity property of positive solutions in this definition. We should also point out why ψ maps $\mathbb{R}^L_+$ to its boundary. Indeed, this is obvious for case (i) of the definition and when $\alpha \in \partial\mathbb{R}^L_+$. To see this for case (ii), observe that if we send $r \longrightarrow \infty$ in (3.1), basic ODE or elliptic theory indicates that $0 = F(\psi(\alpha))$. From this, the non-degeneracy conditions imply that $\psi(\alpha) \in \partial\mathbb{R}^L_+$.

Proof of Theorem 3. We divide the proof into three main steps. The first two steps will show that ψ maps hyperplanes in the positive quadrant onto the wall provided that ψ is continuous. Then, the existence of entire bound state solutions will follow from this surjectivity property and the non-existence of solutions for (Local). Finally, the continuity of the target map is then shown in the remaining step.

Step 1: We claim that $\psi : \mathbb{R}^L_+ \longrightarrow \partial\mathbb{R}^L_+$ is continuous, but we postpone the proof of this until Step 3 in order to better convey the ideas of our method. With this continuity assumption, fix any $a > 0$ and set

$$P_a := \left\{ \alpha \in \mathbb{R}^L_+ \mid \sum_{i=1}^{L} \alpha_i = a \right\} \text{ and } Q_a := \left\{ \alpha \in \partial\mathbb{R}^L_+ \mid \sum_{i=1}^{L} \alpha_i \leq a \right\}.$$

We show that $\psi : P_a \longrightarrow Q_a$ is an onto map. To see this, first observe that ψ maps P_a into Q_a by the non-increasing property of solutions. Now define the homeomorphism $\varphi : Q_a \longrightarrow P_a$ where

$$\varphi(\alpha) = \alpha + \frac{1}{L}\left(a - \sum_{i=1}^{L} \alpha_i \right)(1, 1, \cdots, 1)$$

whose inverse is given by

$$\varphi^{-1}(\alpha) = \alpha - \left(\min_{i=1,\cdots,L} \alpha_i \right)(1, 1, \cdots, 1).$$

Set $G = \varphi \circ \psi : P_a \longrightarrow P_a$. Indeed, G is continuous on P_a and is equivalent to the identity map on the boundary of P_a. Then from elementary degree theory (see [23]), the index of the map satisfies $degree(G, P_a, \alpha) = degree(Identity, P_a, \alpha) = 1$ for each interior point $\alpha \in int(P_a)$. Hence, this ensures that G is onto and thus ψ is onto.

Step 2: Now, the surjectivity of $\psi : P_a \longrightarrow Q_a$ from Step 1 implies that we can find a positive element $\alpha_a \in int(P_a)$ such that $\psi(\alpha_a) = 0$. If $u(r, \alpha_a)$ denotes the solution of (3.1) with the initial value α_a, we claim that this solution never touches the wall, i.e.,

$$u(r, \alpha_a) > 0 \text{ for all } r > 0. \tag{3.2}$$

On the contrary, if we assume there was such a smallest r_0 such that $u_{i_0}(r_0, \alpha_a) = 0$ for some $i_0 \in \{1, 2, \ldots, L\}$, then this would imply that $u(r_0, \alpha_a) = \psi(\alpha_a) = 0$ and $u(x) = u(|x|, \alpha_a)$ would be a radially symmetric solution of (Local) with $R = r_0$. This is impossible and thus assertion (3.2) holds. In addition, notice that we have that $u(x) \longrightarrow \psi(\alpha_a) = 0$ uniformly as $|x| \longrightarrow \infty$. Hence, $u(x) = u(|x|, \alpha_a)$ is indeed a solution of (Global).

Step 3: It only remains to show that $\psi : \mathbb{R}^L_+ \longrightarrow \partial\mathbb{R}^L_+$ is continuous.

Fix an $\epsilon > 0$ and choose any $\overline{\alpha} \in \mathbb{R}^L_+$. To prove ψ is continuous at $\overline{\alpha}$, there are three cases to consider: (a) when $\overline{\alpha} \in \partial\mathbb{R}^L_+$; (b) when $\overline{\alpha} \in int(\mathbb{R}^L_+)$ and the solution to (3.1) with initial value $\overline{\alpha}$ touches the wall; and (c) when $\overline{\alpha} \in int(\mathbb{R}^L_+)$ and the solution to (3.1) with initial value $\overline{\alpha}$ never touches the wall.

Case (a): The continuity of ψ at $\overline{\alpha} = 0$ is trivial and follows directly from the non-increasing property of solutions, since $|\psi(\alpha) - \psi(\overline{\alpha})| = |\psi(\alpha)| \leq |\alpha| \longrightarrow 0$ as $\alpha \longrightarrow \overline{\alpha} = 0$. Therefore, we assume $\overline{\alpha}$ is a non-zero boundary point. In fact, from the non-degeneracy conditions, we can assume that $\overline{\alpha}_1 = \overline{\alpha}_2 = \cdots = \overline{\alpha}_j = 0$ and $\overline{\alpha}_{j+1}, \dots, \overline{\alpha}_L > 0$ for some $j \neq L$.

The first j components of ψ are continuous at $\overline{\alpha}$ since

$$|\psi_k(\alpha) - \psi_k(\overline{\alpha})| \leq |\alpha_k| \longrightarrow 0 \text{ as } |\alpha - \overline{\alpha}| \longrightarrow 0 \text{ for } k = 1, 2, \dots, j.$$

The continuity of the remaining components will follow from the non-degeneracy conditions. Roughly speaking, the idea is to exploit (2.1) to control the larger components with the smaller ones.

First, we observe that if we multiply the ODE in (3.1) by r^{n-1} then integrate, we can see that the solution with initial value $\alpha \in int(\mathbb{R}^L_+)$ obeys the representation

$$\alpha_k - u_k(r,\alpha) = \int_0^r \frac{1}{\xi^{n-1}} \int_0^\xi \eta^{n-1} f_k(u(\eta,\alpha))\, d\eta\, d\xi =: G_k(r,\alpha) \text{ for all } k. \tag{3.3}$$

Now set $2m := \min_{j+1 \leq k \leq L} \overline{\alpha}_k$, $M := |\overline{\alpha}|$, and choose a suitably small $\delta > 0$ satisfying

$$\delta \leq \min\{m/2, \epsilon/2\} \cdot (1 + C_{m,M})^{-1}, \tag{3.4}$$

such that $|\alpha - \overline{\alpha}| < \delta$. We claim that for r in $[0, r_0]$ (or in $[0,\infty)$ if the solution never touches the wall), there holds

$$\begin{cases} u_k(r,\alpha) \leq m & \text{for } k = 1, \dots, j, \\ u_k(r,\alpha) > m & \text{for } k = j+1, \dots, L. \end{cases} \tag{3.5}$$

Once we verify this assertion, (2.1) with (3.3) would then imply that

$$\begin{aligned} |\psi_k(\overline{\alpha}) - \psi_k(\alpha)| &= |\overline{\alpha}_k - \psi_k(\alpha)| \leq |\overline{\alpha}_k - \alpha_k| + |G_k(r_0,\alpha)| \\ &< \delta + C_{m,M}\Big\{\frac{1}{L}\sum_{i=1}^j G_i(r_0,\alpha)\Big\} < (1 + C_{m,M})\delta < \epsilon \end{aligned}$$

for $k = j+1, \dots, L$, and thus we arrive at the desired result. Hence, it remains to verify that (3.5) holds. Indeed, for $k = 1, 2, \dots, j$, $u_k(r,\alpha) \leq m$ by the non-increasing property of solutions. To show boundedness of u_k from below by m for $k = j+1, j+2, \dots, L$, we argue by contradiction. That is, if we assume that there exist $k_0 \in \{j+1, \dots, L\}$ with the shortest bounded interval $[0, r_{\alpha,k_0})$ such that $u_{k_0}(r,\alpha) > m$ for all $r \in [0, r_{\alpha,k_0})$ and $u_{k_0}(r_{\alpha,k_0}, \alpha) = m$, then (2.1), (3.3) and (3.4) imply

$$\begin{aligned} u_{k_0}(r,\alpha) = \alpha_{k_0} - G_{k_0}(r,\alpha) &\geq \overline{\alpha}_{k_0} - \delta - C_{m,M}\Big\{\frac{1}{L}\sum_{i=1}^j G_i(r,\alpha)\Big\} \geq \overline{\alpha}_{k_0} - \delta - C_{m,M}\delta \\ &\geq 2m - \delta(1 + C_{m,M}) \geq 2m - m/2 = (3/2)m \quad \text{for } r \in [0, r_{\alpha,k_0}). \end{aligned}$$

But this is impossible, and we complete the proof for this case.

Case (b): As described earlier, sending $r \longrightarrow \infty$ in (3.1) yields $F(\psi(\overline{\alpha})) = 0$, which further implies that $\psi(\overline{\alpha}) \in \partial\mathbb{R}^L_+$. Without loss of generality, we can assume

$$\psi_1(\overline{\alpha}) = \psi_2(\overline{\alpha}) = \cdots = \psi_j(\overline{\alpha}) = 0 \text{ and } \psi_{j+1}(\overline{\alpha}), \dots, \psi_L(\overline{\alpha}) > 0 \text{ for some } j \neq L,$$

since the case $\psi(\overline{\alpha}) = 0$ can be treated similarly but is much simpler (see [29]). Choose a suitably small $\delta \in (0, \epsilon/[6(1 + C_{m,M})])$ and a large $R > 0$ such that for $|\alpha - \overline{\alpha}| < \delta$,

$$|u(R, \overline{\alpha}) - \psi(\overline{\alpha})| < \epsilon/3, \tag{3.6}$$

$$u(r, \alpha) > 0 \text{ on } [0, R] \text{ and } |u(R, \alpha) - u(R, \overline{\alpha})| < \epsilon/3. \tag{3.7}$$

Then for $k = 1, 2, \dots, j$, we have that

$$|\psi_k(\alpha) - \psi_k(\overline{\alpha})| = |\psi_k(\alpha)| \leq |u_k(R, \alpha)| \leq |u_k(R, \alpha) - u_k(R, \overline{\alpha})| + |u_k(R, \overline{\alpha})| < \epsilon.$$

We now show the remaining larger components of ψ are continuous at $\overline{\alpha}$ by adopting ideas from Case (a). Actually, it is easy to see that it is enough to show that

$$|u_k(r, \alpha) - \psi_k(\overline{\alpha})| < \epsilon \text{ for all } r \geq R \text{ and } k = j + 1, j + 2, \dots, L.$$

Indeed, if we set $2m := \min_{j+1 \leq i \leq L} u_i(R, \overline{\alpha})$ and $M = |\overline{\alpha}|$, then we can still show (3.5) holds but for $r \geq R$. Then, (2.1) with (3.3) and (3.7) imply that

$$\begin{aligned} |u_k(r, \alpha) - u_k(R, \overline{\alpha})| &\leq |\alpha_k - \overline{\alpha}_k| + |G_k(r, \alpha) - G_k(R, \overline{\alpha})| \\ &\leq \delta + |G_k(R, \alpha) - G_k(R, \overline{\alpha})| + |G_k(r, \alpha) - G_k(R, \alpha)| \\ &\leq 2(1 + C_{m,M})\delta + \epsilon/3 < (2/3)\epsilon. \end{aligned}$$

Hence, combining this with (3.6) yields

$$|u_k(r, \alpha) - \psi_k(\overline{\alpha})| \leq |u_k(r, \alpha) - u_k(R, \overline{\alpha})| + |u_k(R, \overline{\alpha}) - \psi_k(\overline{\alpha})| < \epsilon \text{ for } r \geq R.$$

Case (c): Since the source terms f_i are non-negative, a direct calculation shows that $u'_{i_0}(r_0, \overline{\alpha}) < 0$. This transversality condition along with ODE stability imply that for α sufficiently close to $\overline{\alpha}$, the solution to this perturbed initial value problem must touch the wall and $\psi(\alpha)$ must be near $\psi(\overline{\alpha})$. This completes the proof of the continuity of ψ at $\overline{\alpha}$. $\square$

4. Non-existence results and the proofs of Theorems 1 and 2

Proposition 1. *There hold the following.*

(i) Let $s, t, p, q > 1$. The boundary value problem,

$$\begin{cases} -\Delta u = u^s v^q, & u > 0 \quad \text{in } B_R(0), \\ -\Delta v = v^t u^p, & v > 0 \quad \text{in } B_R(0), \\ u = v = 0 & \quad\quad\quad \text{on } \partial B_R(0), \end{cases}$$

has no solution for any $R > 0$ if either

$$\min\{s + q, t + p\} \geq \frac{n + 2}{n - 2},$$

or

$$\frac{1}{1 + q} + \frac{1}{1 + p} \leq \frac{n - 2}{n}.$$

(ii) If $\min\{s,t,p,q\} \geq \frac{n+2}{n-2}$*, then the system*

$$\begin{cases} -\Delta u = u^s + v^q, \quad u > 0 & \text{in } B_R(0), \\ -\Delta v = v^t + u^p, \quad v > 0 & \text{in } B_R(0), \\ \quad u = v = 0 & \text{on } \partial B_R(0), \end{cases}$$

has no solution for any $R > 0$*.*

Part (i) can be found in [26] (see Theorem 4) and [22] (see Proposition 4.1) and part (ii) can be found in [30] (see Theorem 5.6).

Proofs of Theorems 1 and 2. (a) It is clear that the existence results of Theorems 1 and 2 follow directly from Theorem 3 and Proposition 1.

(b) The result on the symmetry of components of Theorem 1 follows from simple ODE arguments much like those found in Section 3 of [14], but we include the proof for completeness. More precisely, if (u,v) is a radially symmetric solution of (1.2), then it suffices to show that $u(0) = v(0)$, since basic ODE uniqueness theory implies $u(r) = v(r)$ for all $r \geq 0$. Assume otherwise. On one hand, if we assume that $u(0) > v(0)$, then there is a maximal interval $[0, R_0)$ for some $R_0 \in (0,\infty]$ such that $u(r) > v(r)$ for $r \in [0, R_0)$. Thus, condition (1.8) yields $u^{p-s}(r) > v^{q-t}(r)$ in $[0, R_0)$ and so

$$v^t(r)u^p(r) > u^s(r)v^q(r) \text{ for } r \in [0, R_0). \tag{4.1}$$

As before, we calculate that for $r \geq 0$,

$$u(r) = u(0) - \int_0^r \frac{1}{\xi^{n-1}} \int_0^\xi \eta^{n-1} u^s(\eta) v^q(\eta)\, d\eta\, d\xi =: u(0) - G_1(r),$$
$$v(r) = v(0) - \int_0^r \frac{1}{\xi^{n-1}} \int_0^\xi \eta^{n-1} v^t(\eta) u^p(\eta)\, d\eta\, d\xi =: v(0) - G_2(r),$$

and from (4.1) we get that $G_2(R_0) \geq G_1(R_0)$. In addition, $u(R_0) = v(R_0)$ since $u, v \longrightarrow 0$ as $|x| \longrightarrow \infty$. Thus

$$0 > v(0) - u(0) = v(R_0) - u(R_0) + G_2(R_0) - G_1(R_0) = G_2(R_0) - G_1(R_0) \geq 0$$

and we deduce a contradiction. Likewise, if we assume $u(0) < v(0)$, we can mimic the same argument above to arrive at the desired contradiction. Hence, we conclude that $u(0) = v(0)$ and this completes the proof of the theorem. ☐

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